MEASURING FRACTALS BY INFINITE AND INFINITESIMAL NUMBERS

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Abstract. Traditional mathematical tools used for analysis of fractals allow one to distinguish results of self-similarity processes after a finite number of iterations. For example, the result of procedure of construction of Cantor's set after two steps is different from that obtained after three steps. However, we are not able to make such a distinction at infinity. It is shown in this paper that infinite and infinitesimal numbers proposed recently allow one to measure results of fractal processes at different iterations at infinity too. First, the new technique is used to measure at infinity sets being results of Cantor's procedure. Second, it is applied to calculate the lengths of polygonal geometric spirals at different points of infinity.

1. INTRODUCTION

During last decades fractals have been intensively studied and applied in various fields (see, for instance, [4, 11, 5, 7, 12, 20]). However, their mathematical analysis (except, of course, a very well developed theory of fractal dimensions) very often continues to have mainly a qualitative character and there are no many tools for a quantitative analysis of their behavior after execution of infinitely many steps of a self-similarity process of construction.

Usually, we can measure fractals in a way and can give certain numerical answers to questions regarding fractals only if a finite number of steps in the procedure of their construction has been executed. The same questions can remain without any answer if we consider execution of an infinite number of steps. For example, let us consider the famous fractal construction – Cantor's set (see Fig. 1). If a finite number of steps, n, has been done constructing Cantor's set, then we are able to describe numerically the set being the result of this operation. It will have 2^n intervals having the length

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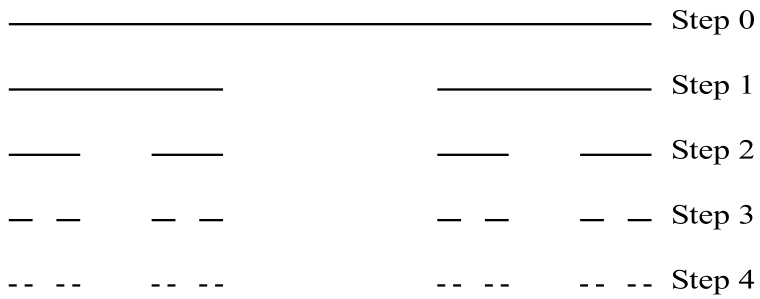

FIGURE 1. Cantor's construction.

$\frac{1}{3^n}$ each. Obviously, the set obtained after $n+1$ iterations will be different and we also are able to measure the lengths of the intervals forming the second set. It will have 2^{n+1} intervals having the length $\frac{1}{3^{n+1}}$ each. The situation changes drastically in the limit because we are not able to distinguish results of n and $n+1$ steps of the construction if n is infinite.

We also are not able to distinguish at infinity the results of the following two processes that both use Cantor's construction but start from different positions. The first one is the usual Cantor's set and it starts from the interval $[0,1]$, the second starts from the couple of intervals $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$. In spite of the fact that for any given finite number of steps, n, the results of the constructions will be different for these two processes, we have no tools to distinguish and, therefore, to measure them at infinity.

Another class of fractal objects that defies length measurement are spirals that fascinated mathematicians throughout the ages (see, e.g., [12]). Let us consider two kinds of polygonal spirals shown in Figs. 2 and 3. Both of them are geometric polygonal spirals related to geometric sequences. The spiral shown in Fig. 2 is constructed as follows. The unit interval is our initial piece and we draw it vertically from bottom to top. At the end we make a right turn and draw the unit interval again from left to right. Then we draw the interval having a length $q<1$ by continuation in the same direction from left to right. At the end we make another right turn and draw again the same interval having the length q from top to bottom. At the end of this line we draw the interval with the length q^2 and continue using the same principle. Fig. 3 shows the same construction for $q>1$. Evidently, for $q=1$ we obtain just a square.

If we try to calculate the length of geometric polygonal spirals, we obtain immediately that it is equal to

$$S = 2(1+q+q^2+q^3+\ldots) = 2\sum_{i=0}^{\infty} q^i, \tag{1.1}$$

which is a geometric series and, therefore, its limiting length for $q<1$ is $2/(1-q)$, i.e., a finite value different for each given q. Then, for all $q>1$ traditional analysis

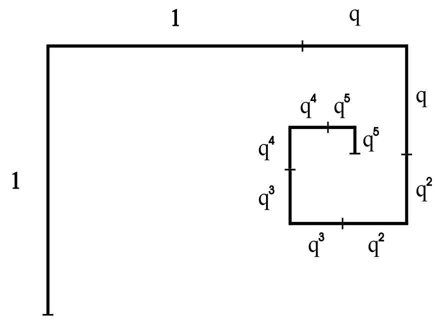

FIGURE 2. The first construction steps of a polygonal geometric spiral with $q < 1$.

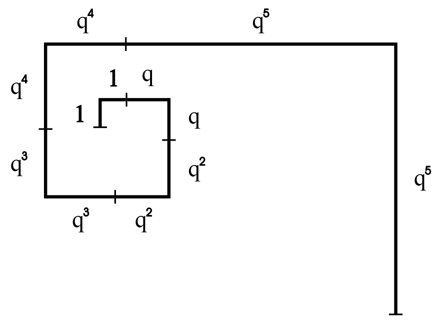

FIGURE 3. The first construction steps of a polygonal geometric spiral with $q > 1$.

tells us that the spiral has the infinite length, i.e., we are not able to distinguish the spirals in dependence of the value of q.

In this paper, we show how a recently developed approach (see [8, 15, 16, 17, 18, 19]) that allows one to write down infinite and infinitesimal numbers and to execute arithmetical operations with them can be used for measuring fractals at infinity. Particularly, the lengths of intervals of Cantor's set and the lengths of spirals from Figs. 2 and 3 for any q will be calculated.

The rest of the paper is organized as follows. Section 2 introduces the new methodology and Section 3 describes a general framework allowing one to express by a finite number of symbols not only finite but infinite and infinitesimal numbers, too. Section 4 describes how infinite and infinitesimal numbers can be used for measuring fractal objects. Finally, Section 5 contains a brief conclusion.

2. METHODOLOGY

Usually, when mathematicians deal with infinite objects (sets or processes) it is supposed that human beings are able to execute certain operations infinitely many times (see [1, 2, 3, 10, 14]). For example, in a fixed numeral system it is possible to write down a numeral[1] with *any* number of digits. However, this supposition is an abstraction (courageously declared by constructivists in [9]) because we live in a finite world and all human beings and/or computers finish operations they have started.

The new computational paradigm introduced in [16, 17, 18, 19] does not use this abstraction and, therefore, is closer to the world of practical calculations than traditional approaches. Its strong computational character is enforced also by the fact that the first simulator of the Infinity Computer able to work with infinite, finite, and infinitesimal numbers introduced in [16, 17, 18, 19] has been already realized (see [8, 15]).

In order to introduce the new methodology, let us consider a study published in *Science* by Peter Gordon (see [6]) where he describes a primitive tribe living in Amazonia - Pirahã - that uses a very simple numeral system for counting: one, two, many. For Pirahã, all quantities bigger than two are just 'many' and such operations as 2+2 and 2+1 give the same result, i.e., 'many'. Using their weak numeral system Pirahã are not able to see, for instance, numbers 3, 4, 5, and 6, to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words nor concepts for that. Moreover, the weakness of their numeral system leads to such results as

$$\text{'many'} + 1 = \text{'many'}, \qquad \text{'many'} + 2 = \text{'many'},$$

which are very familiar to us in the context of views on infinity used in the traditional calculus

$$\infty + 1 = \infty, \qquad \infty + 2 = \infty.$$

This observation leads us to the following idea: *Probably our difficulty in working with infinity is not connected to the nature of infinity but is a result of inadequate numeral systems used to express numbers.*

We start by introducing three postulates that will fix our methodological positions with respect to infinite and infinitesimal quantities and mathematics, in general.

Postulate 1. *We accept that human beings and machines are able to execute only a finite number of operations.*

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[1] We remind that *numeral* is a symbol or group of symbols that represents a *number*. The difference between numerals and numbers is the same as the difference between words and the things they refer to. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols '6', 'six', and 'VI' are different numerals, but they all represent the same number.

Thus, we accept that we shall never be able to give a complete description of infinite processes and sets due to our finite capabilities. Particularly, this means that we accept that we are able to write down only a finite number of symbols to express numbers.

The second postulate that will be adopted is due to the following consideration. In natural sciences, researchers use tools to describe the object of their study and the used instrument influences results of observations. When physicists see a black dot in their microscope they cannot say: The object of observation *is* the black dot. They are obliged to say: the lens used in the microscope allows us to see the black dot and it is not possible to say anything more about the nature of the object of observation until we'll not change the instrument - the lens or the microscope itself - by a more precise one.

Due to Postulate 1, the same happens in mathematics studying natural phenomena, numbers, and objects that can be constructed by using numbers. Numeral systems used to express numbers are among the instruments of observations used by mathematicians. Usage of powerful numeral systems gives possibility to obtain more precise results in mathematics in the same way as usage of a good microscope gives a possibility to obtain more precise results in physics. However, the capabilities of all mathematical tools will be always limited due to Postulate 1.

Postulate 2. *Following natural sciences, we shall not tell* ***what are*** *the mathematical objects we deal with; we just shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.*

Particularly, this means that from our point of view, axiomatic systems do not define mathematical objects but just determine formal rules for operating with certain numerals reflecting some properties of the studied mathematical objects.

After all, we want to treat infinite and infinitesimal numbers in the same manner as we are used to deal with finite ones, i.e., by applying the philosophical principle of Ancient Greeks 'The part is less than the whole'. This principle, in our opinion, very well reflects organization of the world around us but is not incorporated in many traditional infinity theories where it is true only for finite numbers.

Postulate 3. *Following Ancient Greeks, we adopt the principle 'The part is less than the whole' to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite).*

Due to this declared applied statement, such concepts as bijection, numerable and continuum sets, cardinal and ordinal numbers cannot be used in this paper because they belong to the theories working with different assumptions[2]. However, the approach proposed here does not contradict Cantor. In contrast, it evolves his deep ideas regarding existence of different infinite numbers in a more applied way.

[2]As a consequence, the approach used in this paper is different also with respect to non-standard analysis introduced in [14] and built using Cantor's ideas.

Let us start our consideration by studying situations arising in practice when it is necessary to operate with extremely large quantities (see [16] for a detailed discussion). Imagine that we are in a granary and the owner asks us to count how much grain he has inside it. There are a few possibilities of finding an answer to this question. The first one is to count the grain seed by seed. Of course, nobody can do this because the number of seeds is enormous.

To overcome this difficulty, people take sacks, fill them in with seeds, and count the number of sacks. It is important that nobody counts the number of seeds in a sack. At the end of the counting procedure, we shall have a number of sacks completely filled and some remaining seeds that are not sufficient to complete the next sack. At this moment it is possible to return to the seeds and to count the number of remaining seeds that have not been put in sacks (or a number of seeds that it is necessary to add to obtain the last completely full sack).

If the granary is huge and it becomes difficult to count the sacks, then trucks or even big train waggons are used. Of course, we suppose that all sacks contain the same number of seeds, all trucks – the same number of sacks, and all waggons – the same number of trucks. At the end of the counting we obtain a result in the following form: the granary contains 17 waggons, 23 trucks, 2 sacks, and 84 seeds of grain. Note, that if we add, for example, one seed to the granary, we can count it and see that the granary has more grain. If we take out one waggon, we again be able to say how much grain has been subtracted.

Thus, in our example it is necessary to count large quantities. They are finite but it is impossible to count them directly using elementary units of measure, u_0, i.e., seeds, because the quantities expressed in these units would be too large. Therefore, people are forced to behave as if the quantities were infinite.

To solve the problem of 'infinite' quantities, new units of measure, u_1, u_2, and u_3, are introduced (units u_1 – sacks, u_2 – trucks, and u_3 – waggons). The new units have the following important peculiarity: it is not known how many units u_i there are in the unit u_{i+1} (we do not count how many seeds are in a sack, we just *complete* the sack). Every unit u_{i+1} is filled in completely by the units u_i. Thus, we know that all the units u_{i+1} contain a certain number K_i of units u_i but this number, K_i, is unknown. Naturally, it is supposed that K_i is the same for all instances of the units. Thus, numbers that it was impossible to express using only initial units of measure are perfectly expressible if new units are introduced. This key idea of counting by introduction of new units of measure will be used in the paper to deal with infinite quantities.

In order to have a possibility to write down infinite and infinitesimal numbers by a finite number of symbols, we need at least one new numeral expressing an infinite (or an infinitesimal) number corresponding to the chosen infinite unit of measure[3]. Then, it is necessary to propose a new numeral system fixing rules for writing down infinite and infinitesimal numerals and to describe arithmetical operations with them.

3. Infinite and infinitesimal numbers and operations with them

Different numeral systems have been developed by humanity to describe finite numbers. More powerful numeral systems allow us to write down more numerals and, therefore, to express more numbers. A new positional numeral system with infinite radix described in this section evolves the idea of separate count of units with different exponents used in traditional positional systems to the case of infinite and infinitesimal numbers.

The infinite radix of the new system is introduced as the number of elements of the set $\mathbb{N}$ of natural numbers expressed by the numeral ① called *grossone*. This mathematical object is introduced by describing its properties postulated by the *Infinite Unit Axiom* consisting of three parts: Infinity, Identity, and Divisibility (we introduce them soon). This axiom is added to axioms for real numbers similarly to addition of the axiom determining zero to axioms of natural numbers when integer numbers are introduced. This means that it is postulated that associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, existence of inverse elements with respect to addition and multiplication hold for grossone as for finite numbers.

Note that usage of a numeral indicating totality of the elements we deal with is not new in mathematics. It is sufficient to remind the theory of probability where events can be defined in two ways. First, as union of elementary events; second, as a sample space, Ω, of all possible elementary events from where some elementary events have been excluded. Naturally, the second way to define events becomes particularly useful when the sample space consists of infinitely many elementary events.

The Infinite Unit Axiom consists of the following three statements:

***Infinity*:** For any finite natural number n it follows $n <$ ①.
***Identity*:** The following relations link ① to identity elements 0 and 1

$$0 \cdot ① = ① \cdot 0 = 0, \quad ① - ① = 0, \quad \frac{①}{①} = 1, \quad ①^0 = 1, \quad 1^{①} = 1, \quad 0^{①} = 0. \qquad (3.1)$$

***Divisibility*:** For any finite natural number n sets $\mathbb{N}_{k,n}, 1 \leq k \leq n$, being the nth parts of the set, $\mathbb{N}$, of natural numbers have the same number of elements

[3] Note that introduction of a new numeral for expressing infinite and infinitesimal numbers is similar to introduction of the concept of zero and the numeral '0' that in the past have allowed people to develop positional systems being more powerful than numeral systems existing before.

indicated by the numeral $\frac{①}{n}$ where

$$\mathbb{N}_{k,n} = \{k, k+n, k+2n, k+3n, \dots\}, \quad 1 \leq k \leq n, \quad \bigcup_{k=1}^{n} \mathbb{N}_{k,n} = \mathbb{N}. \tag{3.2}$$

Divisibility is based on Postulate 3. Let us illustrate it by three examples. If we take $n = 1$, then $\mathbb{N}_{1,1} = \mathbb{N}$ and Divisibility tells that the set, $\mathbb{N}$, of natural numbers has ① elements. If $n = 2$, we have two sets $\mathbb{N}_{1,2}$ and $\mathbb{N}_{2,2}$ and they have $\frac{①}{2}$ elements each. If $n = 3$, then we have three sets $\mathbb{N}_{1,3}$, $\mathbb{N}_{2,3}$, and $\mathbb{N}_{3,3}$ having $\frac{①}{3}$ elements each.

$$\begin{array}{lllllllllll}
① \to & \mathbb{N} = & \{1, & 2, & 3, & 4, & 5, & 6, & 7, & \dots & \} \\
\frac{①}{2} \nearrow & \mathbb{N}_{1,2} = & \{1, & & 3, & & 5, & & 7, & \dots & \} \\
\frac{①}{2} \searrow & \mathbb{N}_{2,2} = & \{ & 2, & & 4, & & 6, & & \dots & \} \\
\frac{①}{3} \nearrow & \mathbb{N}_{1,3} = & \{1, & & & 4, & & & 7, & \dots & \} \\
\frac{①}{3} \to & \mathbb{N}_{2,3} = & \{ & 2, & & & 5, & & & \dots & \} \\
\frac{①}{3} \searrow & \mathbb{N}_{3,3} = & \{ & & 3, & & & 6, & & \dots & \}
\end{array}$$

Before the introduction of the new positional system let us study some properties of grossone. First of all, as was already mentioned above, it is necessary to remind that ① is not either Cantor's $\aleph_0$ or ω that have been introduced in Cantor's theory on the basis of different assumptions. It will be shown hereinafter that grossone unifies both cardinal and ordinal aspects in the same way as finite numerals unify them. Its role in our infinite arithmetic is similar to the role of the number 1 in the finite arithmetic and it will serve us as the basis for construction of other infinite and infinitesimal numbers.

We start by the following important comment: to introduce $\frac{①}{n}$ we do not try to count elements $k, k+n, k+2n, k+3n, \dots$ In fact, we cannot do this due to the accepted Postulate 1. In contrast, we apply Postulate 3 and state that the number of elements of the nth part of the set, i.e., $\frac{①}{n}$, is n times less than the number of elements of the whole set, i.e., than ①. In terms of our granary example ① can be interpreted as the number of seeds in the sack. Then, if the sack contains ① seeds, its nth part contains $\frac{①}{n}$ seeds. It is worthy to emphasize that, since the numbers $\frac{①}{n}$ have been introduced as numbers of elements of sets $\mathbb{N}_{k,n}$, they are integer.

The introduced numerals $\frac{①}{n}$ and the sets $\mathbb{N}_{k,n}$ allow us immediately to calculate the number of elements of certain infinite sets. For example, due to the introduced axiom, the set

$$\{3, 8, 13, 18, 23, 28, \dots\} = \mathbb{N}_{3,5}$$

and, therefore, has $\frac{①}{5}$ elements. The number of elements of sets being union, intersection, difference, or product of other sets of the type $\mathbb{N}_{k,n}$ is defined in the same way as these operations are defined for finite sets. Thus, we can define the number of elements of sets being results of these operations with finite sets and infinite sets of the type $\mathbb{N}_{k,n}$. For example, the set

$$\{3,8,13,18,23,28,\ldots\}\backslash\{3,23\} = \mathbb{N}_{3,5}\backslash\{3,23\}$$

and, therefore, it has $\frac{①}{5} - 2$ elements.

Other results regarding calculating the number of elements of infinite sets can be found in [16, 19]. Particularly, it is shown that the number of elements of the set, $\mathbb{Z}$, of integers is equal to $2① + 1$ and the number of elements of the set, $\mathbb{Q}$, of different rational numerals is equal to $2①^2 + 1$.

The new numeral ① allows us to write down the set, $\mathbb{N}$, of natural numbers in the form

$$\mathbb{N} = \{1,2,3, \quad \ldots \quad ①-2, \; ①-1, \; ①\} \tag{3.3}$$

because *grossone has been introduced as the number of elements of the set of natural numbers* (similarly, the number 3 is the number of elements of the set $\{1, 2, 3\}$). Thus, grossone is the biggest natural number and infinite numbers

$$\ldots \; ①-3, \; ①-2, \; ①-1 \tag{3.4}$$

less than grossone are also natural numbers as the numbers $1,2,3,\ldots$ They can be viewed both in terms of sets of numbers and in terms of grain. For example, $①-1$ can be interpreted as the number of elements of the set $\mathbb{N}$ from which a number has been excluded. In terms of our granary example $①-1$ can be interpreted as a sack minus one seed.

Note that the set (3.3) is the same set of natural numbers we are used to deal with. Infinite numbers (3.4) also take part of the usual set, $\mathbb{N}$, of natural numbers[4]. The difficulty to accept existence of infinite natural numbers is in the fact that traditional numeral systems did not allow us to see them. In the same way as Pirahã are not able to see, for instance, numbers 3, 4, and 5 using their weak numeral system, traditional numeral systems did not allow us to see infinite natural numbers that we can see now using the new numeral ①.

Postulate 3 and the Infinite Unit Axiom allow us to obtain the following important result: the set $\mathbb{N}$ is not a monoid under addition. In fact, the operation $①+1$ gives us as the result a number grater than ①. Thus, by definition of grossone, $①+1$ does not belong to $\mathbb{N}$ and, therefore, $\mathbb{N}$ is not closed under addition and is not a monoid.

This result also means that adding the Infinite Unit Axiom to the axioms of natural numbers defines the set of *extended natural numbers* indicated as $\widehat{\mathbb{N}}$ and including $\mathbb{N}$

[4] This point is one of the differences with respect to non-standard analysis (see [13, 14]) where infinite numbers are not included in $\mathbb{N}$.

as a proper subset

$$\widehat{\mathbb{N}} = \{1,2,\ldots,①-1,①,①+1,\ldots,①^2-1,①^2,①^2+1,\ldots\}.$$

Again, extended natural numbers grater than grossone can also be interpreted in the terms of sets of numbers. For example, $①+3$ as the number of elements of the set $\mathbb{N}\cup\{a,b,c\}$ where numbers $a,b,c \notin \mathbb{N}$ and $①^2$ as the number of elements of the set

$$C = \{(a_1,a_2) : a_1 \in \mathbb{N}, a_2 \in \mathbb{N}\}.$$

In terms of our granary example $①+3$ can be interpreted as one sack plus three seeds and $①^2$ as a truck.

Analogously, we can consider the set, $\widehat{\mathbb{Z}}$, of *extended integer numbers*

$$\widehat{\mathbb{Z}} = \{\ldots,-①-1,-①,-①+1\ldots,-2,-1,0,1,2,\ldots,①-1,①,①+1,\ldots\}.$$

What can we say now about the number of elements of the sets $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{Z}}$? Our positional numeral system with the radix ① dose not allow us to say anything because it does not contain numerals able to express such numbers (see Postulates 1 and 2). It is necessary to introduce in a way a more powerful numeral system defining new numerals ②, ③, etc. However, in spite of the fact that the numeral system using grossone does not allow us to express the numbers of elements of $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{Z}}$, we can work with those subsets of $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{Z}}$ that can be defined by using numerals written down in our positional numeral system with the radix ①.

We have already started to write down simple infinite numbers and to execute arithmetical operations with them without concentrating our attention upon this question. In general, to express a number C in the new numeral positional system with base ① we subdivide C into groups corresponding to powers of ①:

$$C = c_{p_m}①^{p_m} + \ldots + c_{p_1}①^{p_1} + c_{p_0}①^{p_0} + c_{p_{-1}}①^{p_{-1}} + \ldots + c_{p_{-k}}①^{p_{-k}}. \tag{3.5}$$

Then, the record

$$C = c_{p_m}①^{p_m} \ldots c_{p_1}①^{p_1} c_{p_0}①^{p_0} c_{p_{-1}}①^{p_{-1}} \ldots c_{p_{-k}}①^{p_{-k}} \tag{3.6}$$

represents the number C, where finite numbers c_i are called *infinite grossdigits* and can be both positive and negative; numbers p_i are called *grosspowers* and can be finite, infinite, and infinitesimal (the introduction of infinitesimal numbers will be given soon). The numbers p_i are such that $p_i > 0, p_0 = 0, p_{-i} < 0$ and

$$p_m > p_{m-1} > \ldots p_2 > p_1 > p_{-1} > p_{-2} > \ldots p_{-(k-1)} > p_{-k}.$$

In the traditional positional systems there exists a convention that a digit a_i shows how many powers b^i are present in the number and the radix b is not written explicitly. In the record (3.6) we write $①^{p_i}$ explicitly because in the new numeral positional system the number i in general is not equal to the grosspower p_i. This gives possibility to write, for example, such numbers as $\frac{7}{3}①^4 \, \frac{84}{19}①^{-3.1}$ where $p_1 = 4, p_{-1} = -3.1$. Grossdigits $c_i, -k \leq i \leq m$, can be integer or fractional and expressed by many symbols; in our example, $c_4 = \frac{7}{3}$ and $c_{-3.1} = \frac{84}{19}$.

Finite numbers in this new numeral system are represented by numerals having only one grosspower equal to zero. In fact, if we have a number C such that $m = k = 0$ in representation (3.6), then due to (3.1) we have $C = c_0①^0 = c_0$. Thus, the number C in this case does not contain infinite units and is equal to the grossdigit c_0 which being a conventional finite number can be expressed by any positional system with finite base b (or by another traditional numeral system). It is important to emphasize that the grossdigit c_0 can be integer or fractional and can be expressed by a few symbols in contrast to the traditional positional systems where each digit is integer and is represented by one symbol from the alphabet $\{0,1,2,\ldots,b-1\}$. Thus, the grossdigit c_0 shows how many finite units and/or parts of the finite unit, $1 = ①^0$, there are in the number C.

Infinite numbers in this numeral system are expressed by numerals having at least one grosspower grater than zero. In the following example the left-hand expression presents the way to write down infinite numbers and the right-hand shows how the value of the number is calculated:

$$21.4①^{23}\text{-}1.45①^{3.4}852.1①^{-66.2} = 21.4①^{23} - 1.45①^{3.4} + 852.1①^{-66.2}.$$

If a grossdigit c_{p_i} is equal to 1 then we write $①^{p_i}$ instead of $1①^{p_i}$. Analogously, if power $①^0$ is the lowest in a number then we often use simply the corresponding grossdigit c_0 without $①^0$, for instance, we write $23①^{14}5$ instead of $23①^{14}5①^0$ or 8 instead of $8①^0$.

Infinitesimal numbers are represented by numerals having only negative grosspowers. The simplest number from this group is $①^{-1} = \frac{1}{①}$ being the inverse element with respect to multiplication for ①:

$$\frac{1}{①} \cdot ① = ① \cdot \frac{1}{①} = 1. \tag{3.7}$$

Note that all infinitesimals are not equal to zero. Particularly, $\frac{1}{①} > 0$ because $1 > 0$ and $① > 0$. It has a clear interpretation in our granary example. Namely, if we have a sack and it contains ① seeds then one sack divided by ① is equal to one seed. Vice versa, one seed, i.e., $\frac{1}{①}$, multiplied by the number of seeds in the sack, ①, gives one sack of seeds.

Let us now introduce arithmetical operations for infinite, infinitesimal, and finite numbers (see [16] for a detailed discussion and examples). The numbers A, B, and their sum C are represented in the record of the type

$$A = \sum_{i=1}^{K} a_{k_i}①^{k_i}, \qquad B = \sum_{j=1}^{M} b_{m_j}①^{m_j}, \qquad C = \sum_{i=1}^{L} c_{l_i}①^{l_i}. \tag{3.8}$$

The operation of *addition* of two given infinite numbers A and B returns as the result an infinite number C constructed as follows (the operation of subtraction is a direct consequence of that of addition and is thus omitted). Then the result C is constructed by including in it all items $a_{k_i}①^{k_i}$ from A such that $k_i \neq m_j, 1 \leq j \leq M$, and all items $b_{m_j}①^{m_j}$ from B such that $m_j \neq k_i, 1 \leq i \leq K$. If in A and B there are items such that $k_i = m_j$ for some i and j then this grosspower k_i is included in C with the grossdigit $b_{k_i} + a_{k_i}$, i.e., as $(b_{k_i} + a_{k_i})①^{k_i}$. It can be seen from this definition that the introduced operation enjoys the usual properties of commutativity and associativity due to definition of grossdigits and the fact that addition for each grosspower of ① is executed separately.

The operation of *multiplication* of two given infinite numbers A and B from (3.8) returns as the result the infinite number C constructed as follows.

$$C = \sum_{j=1}^{M} C_j, \quad C_j = b_{m_j}①^{m_j} \cdot A = \sum_{i=1}^{K} a_{k_i} b_{m_j} ①^{k_i + m_j}, \quad 1 \leq j \leq M. \tag{3.9}$$

Similarly to addition, the introduced multiplication is commutative and associative. It is easy to show that the distributive property is also valid for these operations.

In the operation of *division* of a given infinite number C by an infinite number B we obtain an infinite number A and a reminder R that can be also equal to zero, i.e., $C = A \cdot B + R$.

The number A is constructed as follows. The numbers B and C are represented in the form (3.8). The first grossdigit a_{k_K} and the corresponding maximal exponent k_K are established from the equalities

$$a_{k_K} = c_{l_L} / b_{m_M}, \quad k_K = l_L - m_M. \tag{3.10}$$

Then the first partial reminder R_1 is calculated as

$$R_1 = C - a_{k_K}①^{k_K} \cdot B. \tag{3.11}$$

If $R_1 \neq 0$ then the number C is substituted by R_1 and the process is repeated by a complete analogy. The grossdigit $a_{k_{K-i}}$, the corresponding grosspower k_{K-i} and the partial reminder R_{i+1} are computed by formulae (3.12) and (3.13) obtained from (3.10) and (3.11) as follows: l_L and c_{l_L} are substituted by the highest grosspower n_i and the corresponding grossdigit r_{n_i} of the partial reminder R_i that in its turn substitutes C:

$$a_{k_{K-i}} = r_{n_i} / b_{m_M}, \quad k_{K-i} = n_i - m_M. \tag{3.12}$$

$$R_{i+1} = R_i - a_{k_{K-i}}①^{k_{K-i}} \cdot B, \quad i \geq 1. \tag{3.13}$$

The process stops when a partial reminder equal to zero is found (this means that the final reminder $R = 0$) or when a required accuracy of the result is reached.

4. Measuring objects being results of self-similarity processes at infinity

We start by proving the following important result: the number of elements of any infinite sequence is less or equal to ①. To demonstrate this we need to recall the definition of the infinite sequence: 'An infinite sequence $\{a_n\}, a_n \in A$ for all $n \in \mathbb{N}$, is a function having as the domain the set of natural numbers, $\mathbb{N}$, and as the codomain a set $A \subseteq \mathbb{R}$'.

We have postulated in the Infinite Unit Axiom that the set $\mathbb{N}$ has ① elements. Thus, due to the sequence definition given above, any sequence having $\mathbb{N}$ as the domain has ① elements.

One of the immediate consequences of the understanding of this result is that any process can have at maximum ① elements[5]. For example, if we consider the set, $\widehat{\mathbb{Z}}$, of extended integer numbers then starting from the number 1, it is possible to arrive at maximum to ①

$$\ldots,-3,-2,-1,0,\underbrace{1,2,3,4,\ \ldots\ ①-2,\ ①-1,①}_{①},①+1,①+2,①+3,\ldots \quad (4.1)$$

Starting from 0 it is possible to arrive at maximum to $① - 1$

$$\ldots,-3,-2,-1,\underbrace{0,1,2,3,4,\ \ldots\ ①-2,\ ①-1}_{①},①,①+1,①+2,①+3,\ldots \quad (4.2)$$

Starting from -1 it is possible to to arrive at maximum to $① - 2$

$$\ldots,-3,-2,\underbrace{-1,0,1,2,3,4,\ \ldots\ ①-2}_{①},\ ①-1,①,①+1,①+2,①+3,\ldots \quad (4.3)$$

Of course, since we have postulated that our possibilities to express numerals are finite, it depends on the chosen numeral system which numbers among ① members of these processes we can observe.

In order to be able to measure fractals at infinity, we should reconsider the theory of divergent series from the new viewpoint introduced in the previous sections. The introduced numeral system allows us to express not only different finite numbers but also different infinite numbers. Therefore, due to Postulate 3, we should explicitly indicate the number of items in all sums independently on the fact whether this number is finite or infinite. Due to Postulate 2, we shall be able to calculate the sum if its items, the number of items, and the result are expressible in the numeral system used for calculations. It is important to notice that even though a sequence cannot have more than ① elements, the number of items in a sum can be greater than grossone because the process of summing up not necessary should be executed by a sequential adding items.

For instance, let us consider two infinite series

$$S_1 = 2+2+2+2+2+\ldots \qquad S_2 = 1+2+1+2+1+2+1\ldots$$

[5] This observation has a deep relation to the Axiom of Choice. The Infinite Unit Axiom postulates that any process can have at maximum ① elements, thus the process of choice too and, as a consequence, it is not possible to choose more than ① elements from a set. This observation also emphasizes the fact that the parallel computational paradigm is significantly different with respect to the sequential one because p parallel processes can choose $p①$ elements from a set.

The traditional analysis gives us a very poor answer that both of them diverge to infinity. Such operations as $S_1 - S_2$ or $\frac{S_1}{S_2}$ are not defined. From the new point of view, the sums S_1 and S_2 can be calculated but it is necessary to indicate explicitly the number of items in both sums.

Suppose that the sum S_1 has m items and the sum S_2 has n items:

$$S_1(m) = \underbrace{2+2+2+\ldots+2}_{m}, \qquad S_2(n) = \underbrace{1+2+1+2+1+2+1+\ldots}_{n}.$$

Then $S_1(m) = 2m$ and

$$S_2(n) = \underbrace{1+2+1+2+1+2+1+\ldots}_{n} = \begin{cases} k+2k = 3k, & \text{if } n = 2k, \\ k+2k+1 = 3k+1, & \text{if } n = 2k+1, \end{cases}$$

and giving numerical values (finite or infinite) to m and n we obtain numerical values for results of the sums. If, for instance, $m = n = 3①$ then we obtain $S_1(3①) = 6①$, $S_2(3①) = 4.5①$ because ① is even (since, due to the Infinite Unit Axiom, $\frac{①}{2}$ is integer) and

$$S_2(3①) - S_1(3①) = -1.5① < 0.$$

If $m = ①$ and $n = 3① + 1$ we obtain $S_1(①) = 2①$, $S_2(3① + 1) = 4.5① + 1$ and it follows

$$S_2(3①+1) - S_1(①) = 2.5①+1 > 0.$$

If $m = 3①$ and $n = 4①$ we obtain $S_1(3①) = 6①$, $S_2(4①) = 6①$ and it follows

$$S_2(4①) - S_1(3①) = 0.$$

Analogously, the expression $\frac{S_1(k)}{S_2(n)}$ can be calculated.

Let us return now to Cantor's construction and remind that if a finite number of steps, n, has been executed in Cantor's construction starting from the interval $[0,1]$ then we are able to describe numerically the set being the result of this operation. It will have 2^n intervals having the length $\frac{1}{3^n}$ each. Obviously, the set obtained after $n+1$ iterations will be different and we also are able to measure the lengths of the intervals forming the second set. It will have 2^{n+1} intervals having the length $\frac{1}{3^{n+1}}$ each. The situation changes drastically in the limit because traditional approaches are not able to distinguish results of n and $n+1$ steps of the construction if n is infinite. Now, we can do it using the introduced infinite and infinitesimal numbers.

Since the construction of Cantor's set is a process, it cannot contain more then ① steps (see discussion related to the example (4.1)-(4.3)). Thus, if we start the process from the interval $[0,1]$, after ① steps Cantor's set consists of $2^{①}$ intervals and their total length, L_n, is expressed in infinitesimals: $L(①) = (\frac{2}{3})^{①}$, i.e., the set has a well defined infinite number of intervals and each of them has the infinitesimal length equal to $3^{-①}$. Analogously, after $① - 1$ steps Cantor's set consists of $2^{①-1}$ intervals and their total length is expressed in infinitesimals: $L(①) = (\frac{2}{3})^{①-1}$. Thus, the length

L_n for any (finite or infinite) number of steps, n, where $1 \leq n \leq ①$ and is expressible in the chosen numeral system can be calculated.

It is important to notice here that (again due to the limitation illustrated by the example (4.1)-(4.3)) it is not possible to count one by one all the intervals at Cantor's set if their number is superior to ①. For instance, after ① steps it has $2^{①}$ intervals and they cannot be counted one by one because $2^{①} > ①$ and any process (including that of the sequential counting) cannot have more that ① steps.

Let us consider now two processes that both use Cantor's construction but start from different initial conditions. Traditional approaches do not allow us to distinguish them at infinity in spite of the fact that for any given finite number of steps, n, the results of the constructions are different and can be calculated. Using the new approach we are able to study the processes numerically also at infinity. For example, if the first process is the usual Cantor's set and it starts from the interval $[0,1]$ and the second one starts from the couple of intervals $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$ then after $\frac{①}{2}$ steps the result of the first process will be the set consisting of $2^{\frac{①}{2}}$ intervals and its length $L(\frac{①}{2}) = (\frac{2}{3})^{\frac{①}{2}}$. The second set after $\frac{①}{2}$ steps will consists of $2^{\frac{①}{2}+1}$ intervals and its length $L(\frac{①}{2}+1) = (\frac{2}{3})^{\frac{①}{2}+1}$. Naturally, it becomes possible to measure by a complete analogy other classical fractals such as the Koch Curve, the Sierpinski Carpet, etc.

In order to be able to calculate lengths of geometric polygonal spirals from Figs. 2 and 3 at infinity, we need to consider (see (1.1)) the geometric series $\sum_{i=0}^{\infty} q^i$ from the new viewpoint. Traditional analysis proves that it converges to $\frac{1}{1-q}$ for q such that $-1 < q < 1$. We are able to give a more precise answer for *all* values of q. Due to Postulate 3, to do this we should fix the number of items in the sum. If we suppose that it contains n items then

$$Q_n = \sum_{i=0}^{n} q^i = 1 + q + q^2 + \ldots + q^n. \tag{4.4}$$

By multiplying the left hand and the right hand parts of this equality by q and by subtracting the result from (4.4) we obtain

$$Q_n - qQ_n = 1 - q^{n+1}$$

and, as a consequence, for all $q \neq 1$ the formula

$$Q_n = \frac{1 - q^{n+1}}{1 - q} \tag{4.5}$$

holds for finite and infinite n. Thus, the possibility to express infinite and infinitesimal numbers allows us to take into account infinite n too and the value q^{n+1} being infinitesimal for a finite $q < 1$ and infinite for $q > 1$. Moreover, we can calculate Q_n

for $q = 1$ also because in this case we have just

$$Q_n = \underbrace{1+1+1+\ldots+1}_{n+1} = n+1.$$

As the first example, let us consider a spiral with $q = \frac{1}{3}$. Traditional analysis tells us that the length of the spiral is equal to

$$S = 2\sum_{i=0}^{\infty}\frac{1}{3^i} = \frac{2}{1-\frac{1}{3}} = 3.$$

By using the new computational paradigm we are able to give a more precise answer because we are able to distinguish different infinite and infinitesimal numbers and, as a consequence, we can speak not only about tendencies at infinity, as traditional analysis does, but give precise numerical answers. Thus, if we know how many steps, n, have been executed during the process of the construction of the spiral, we can calculate its length S_n for finite and infinite n using formulae (1.1), (4.4), and (4.5): $S_n = 2Q_n$. For example, if ①-1 steps have been executed, the spiral will have the length

$$S_{①-1} = 2(1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^{①-1}}) = 2\cdot\frac{1-\frac{1}{3^{①}}}{1-\frac{1}{3}} = 3-\frac{1}{3^{①-1}}. \qquad (4.6)$$

We can see from this formula that the new numeral system allows us to distinguish and to measure the infinitesimal part of the length of the spiral, $\frac{1}{3^{①-1}}$, that was invisible for traditional numeral systems. Analogously, if $n =$ ①, the length of the spiral is equal to

$$S_{①} = 2(1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^{①-1}}+\frac{1}{3^{①}}) = 2\cdot\frac{1-\frac{1}{3^{①+1}}}{1-\frac{1}{3}} = 3-\frac{1}{3^{①}}. \qquad (4.7)$$

The second spiral has been obtained from the first one by adding two intervals having infinitesimal length $\frac{1}{3^{①}}$. We can obtain this value by subtracting the right part of (4.6) from the right part of (4.7)

$$S_{①} - S_{①-1} = 3-\frac{1}{3^{①}} - (3-\frac{1}{3^{①-1}}) = \frac{2}{3^{①}}.$$

The new numeral system allows us to measure also the lengths of spirals with $q > 1$. For example, from the traditional point of view it is not possible to measure and to compare spirals having $q = 2$ and $q = 3$. We are forced just to say that their lengths are infinite because the series

$$1+2+4+8+\ldots = \sum_{i=0}^{\infty} 2^i, \qquad 1+3+9+27+\ldots = \sum_{i=0}^{\infty} 3^i$$

are divergent. The new approach allows us to measure both spirals and to compare them. Suppose that the spiral with $q = 2$ has been constructed in n steps and the spiral with $q = 3$ in m steps and we want to calculate their lengths A_n and B_m. It follows from (1.1), (4.4), and (4.5) that

$$A_n = 2\frac{1-2^{n+1}}{1-2} = 2^{n+2}-2, \qquad B_m = 2\frac{1-3^{m+1}}{1-3} = 3^{m+1}-1.$$

Both formulae work for finite and infinite values of n and m and allow us to measure the lengths of spirals easily. For example, if $n = m = \frac{①}{2} - 1$, then obviously

$$A_{\frac{①}{2}-1} = 2^{\frac{①}{2}+1}-2, \qquad B_{\frac{①}{2}-1} = 3^{\frac{①}{2}}-1.$$

If in the construction of the first spiral one more step has been executed, i.e., $n = \frac{①}{2}$, and in the construction of the second spiral $m = \frac{①}{2}+1$, i.e., two steps have been added, then

$$A_{\frac{①}{2}} = 2^{\frac{①}{2}+2}-2, \qquad B_{\frac{①}{2}+1} = 3^{\frac{①}{2}+2}-1.$$

We can obtain the same results by direct calculation, i.e., by adding two pieces of the length $2^{\frac{①}{2}}$ to $A_{\frac{①}{2}-1}$ and two times pieces of the length $3^{\frac{①}{2}}$ and $3^{\frac{①}{2}+1}$ to $B_{\frac{①}{2}-1}$

$$A_{\frac{①}{2}-1} + 2\cdot 2^{\frac{①}{2}} = 2^{\frac{①}{2}+1} - 2 + 2^{\frac{①}{2}+1} = 2^{\frac{①}{2}+2}-2,$$

$$B_{\frac{①}{2}-1} + 2(3^{\frac{①}{2}} + 3^{\frac{①}{2}+1}) = 3^{\frac{①}{2}} - 1 + 2(3^{\frac{①}{2}} + 3^{\frac{①}{2}+1}) = 3^{\frac{①}{2}+2}-1.$$

5. A BRIEF CONCLUSION

It has been shown in the paper that the new powerful numeral system allowing us to express not only finite but also infinite and infinitesimal numbers gives a lot of new (in comparison with traditional numeral systems able to express only finite numbers) information about behavior of fractal objects at infinity and can be successfully applied for measuring fractals.

It has been emphasized that the philosophical triad – researcher, object of investigation, and tools used to observe the object – existing in such natural sciences as physics and chemistry exists in mathematics too. Usage of powerful numeral systems gives a possibility to obtain more precise results in mathematics in the same way as usage of a good microscope gives a possibility to obtain more precise results in physics. Infinite and infinitesimal numbers introduced in [15, 16, 17, 18, 19]) allow us to distinguish at infinity different infinite iterations and different fractal objects corresponding to that iterations that were undistinguishable when traditional finite numbers were used.

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